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A NEW COMPLEX VECTOR METHOD FOR BALANCING CHEMICAL EQUATIONS

NOVA KOMPLEKSNA METODA ZA URAVNOTEŽENJE KEMIČNIH ENAČB

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»Just as tensor analysis is the proper language of kinematics, so linear algebra is the proper language of stoichiometry.«
Aris, R.; Mah, R. H. S.; Ind. Eng. Chem. Fundam. 1963, 2, 90.

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In this article, the author discovers a paradox of balancing chemical equations. The many counterexamples illustrate that the considered procedure of balancing chemical equations given in the paper[1] is inconsistent. A new complex vector method for paradox resolution is given too.

Keywords: paradox, balancing, chemical equations, vector analysis.

V članku avtor opisuje paradoks pri uravnoteženju kemijskih reakcij. Več primerov dokazuje, da je procedura uravnoteženja kemijskih reakcij v viru 1 inkonsistentna. Predstavljena je nova kompleksna vektorska metoda za rešitev paradoksa.

Ključne besede: paradoks, uravnoteženje, kemijske reakcije, vektorska analiza

1 INTRODUCTION

Balancing chemical equations is a basic matter of chemistry, if not one of its most important issues, and it plays a main role in its foundation. Indeed, it is a subtle question which deserves considerable attention.

Also, this topic was a magnet for a number of researchers around the world, because the general problem of balancing chemical equations was considered as one of the hardest problems in chemistry, as well as in mathematics.

In chemical literature there are a great number of papers which consider the problem of balancing chemical equations in different *chemical ways*, but all of them offer only some particular procedures for balancing simple chemical equations. Very often, these chemical procedures generate absurd results, because most of them are founded on *presumed chemical principles*, but not on genuine exact principles. And these *presumed chemical principles* generate paradoxes! The balancing chemical equations does not depend on *chemical principles*, it is a mathematical operation which is founded on true mathematical principles.

There is a number of paradoxes in chemistry about balancing chemical equations, and these will be systematized and studied in a special article of higher level by the same author which will appear in the near future. In this article only one paradox in balancing chemical equations, discovered in[1] is considered.

The author would like to emphasize very clearly, that *balancing chemical equations is not chemistry; it is just linear algebra.* Although it is true, that it is not chemistry, it is very important for chemistry! In this particular case the following question comes up: *If balancing chemical equations is not chemistry, then why is it considered in chemistry*? Or maybe more important is this question: *If it is linear algebra, then it should be studied in mathematics, so why bother chemists*?

The author will address the above questions assuming that: *the problem of balancing chemical equations was a multidisciplinary subject and for its solution both mathematicians and chemists are needed. The job of chemists is to perform reactions, while balancing their equations is a job for mathematicians. Mathematics, as a servant to other sciences, is a problem solver, but chemistry is a result user.*

The skepticism about balancing chemical equations by *chemical principles* appeared a long time ago. Let's quote the opinions of three chemists.

In 1926 the American chemist Simons[2] wrote: *The balancing of an equation is a mathematical process and independent of chemistry. The order of steps in the process is as essential as the order of steps in long division and the process is much simpler than is ordinary assumed.* Seventy-one years later, the Dutch chemist Ten Hoor[3], made a similar statement: Balancing the equation of the reaction is a matter of mathematics only.

Now is the right place to paraphrase the criticism of the American chemist Herndon[4]: *The major changes that have taken place over seven decades are substitutions of the terms 'change in oxidation number' and 'algebraic* *method' for the terms 'valence change' and 'method of undetermined coefficients', respectively.* This assertion shows very clearly the picture of chemists' contributions to balancing chemical equations by chemical principles. Probably expressing his satisfaction with the chemists' contributions to balancing chemical equations, Herndon, the editor of the Journal of Chemical Education[4], decided that further discussion *of equation balancing will not appear in the Journal unless it adds something substantively new to what has already appeared*.

In view of the above assertions this question arises: *Are there in chemistry 'chemical principles' capable of offering solutions of the general problem of balancing chemical equations? Maybe more interesting is the question: What are 'chemical principles' – a rhetorical sophism of chemists or their hopes?*

More interesting for us is to give an answer to both questions from a scientific view point. Perhaps, the more appropriate short answer to the first question is: *'Chemical principles' are not defined entities in chemistry, and so this term does not have any meaning. They are not capable to provide solutions of the general problem of balancing chemical equations, because they are founded on an intuitive basis and they represent only a main generator for paradoxes. However, the necessary and sufficient conditions for a complete solution of the general problem of balancing chemical equations lie outside of chemistry, and we must look for them in an amalgamated theory of n-dimensional vector spaces, linear algebra, abstract algebra and topology. It is a very hard problem of the highest level in chemistry and mathematics, which must be considered only on a scientific basis.* Like this should look the answer to the first question.

The answer to the second question will be described in this way: *Actually, 'chemical principles' are a remnant of an old traditional approach in chemistry, when chemists were busy with the verification of their results obtained in chemical experiments. It is true, that until second half of 20th century there was no mathematical method for balancing chemical equations in chemistry, other than Bottomley's algebraic method*[5]. *Chemists balanced simple particular chemical equations using only Johnson's change in oxidation number procedure*[6], *Simons – Waldbauer – Thrun's partial reactions procedure*[2,7] *and other slightly different modifications derived from them. The 'chemical principles' were an assumption of traditional chemists, who thought before the appearance of Jones' problem*[8], *that the solution of the general problem of balancing chemical equations is possible by use of chemical procedures. But, practice showed that the solution of the century old problem is possible only by using contemporary sophisticated mathematical methods*.

These questions require a deeper elaboration than given here, and it will be a main concern of the author in his future research.

2 PRELIMINARIES

The exact statements about balancing chemical equations agree with the following well-known results.

Theorem 1. *Every chemical reaction can be reduced in a matrix equation* $\boldsymbol{Ax} = \boldsymbol{0}$, *where* $\boldsymbol{A}$ *is a reaction matrix, x is a column-vector of the unknown coefficients and* $\boldsymbol{0}$ *is a null column-vector.*

Proof. The proof of this theorem immediately follows from[9].

Remark 2. The coefficients satisfy three basic principles (corresponding to a closed input-output static model[10,11])

- the law of conservation of atoms,
- the law of conservation of mass, and
- the time-independence of the reaction.

Theorem 3. *Every chemical reaction can be presented as a matrix Diophantine equation* $\boldsymbol{Ax} = \boldsymbol{By}$, *where* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are matrices of reactants and products, respectively, and x and y are column-vectors of unknown coefficients.* The proof of this theorem is given in.[12,13]

This theorem is actually the Jones'[8] problem founded by virtue of Crocker's procedure for balancing chemical equations[14], and from this problem the formalization of chemistry began. This problem is a milestone in chemistry and mathematics as well, and just it opened the door in chemistry to enter a new mathematical freshness. Jones[8] with his problem transferred the general problem of balancing chemical equations from the field of chemistry into the field of mathematics and opened way to solve this problem with mathematical methods founded on principles of linear algebra.

Before the appearance of this problem, the approach of balancing chemical equations was intuitive and useful only for some elementary chemical equations. Now, some well-known results from the theory of complex n-dimensional vector spaces[15,16], for resolution of balancing chemical equations will be introduced. Here, by $\mathbb{C}$ is denoted the set of complex numbers and by $\mathbb{C}^n$ is denoted the Euclidian n-dimensional vector space with complex entries.

Definition 4. A vector space over the field $\mathbb{C}$ consists of a set V of objects called vectors for which the axioms for vector addition hold

(A_1) *If* $\boldsymbol{u}, \boldsymbol{v} \in V$, *then* $(\boldsymbol{u} + \boldsymbol{v}) \in V$,

(A_2) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}, \forall \boldsymbol{u}, \boldsymbol{v} \in V$,

(A_3) $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}, \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$,

(A_4) $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u} = \boldsymbol{0} + \boldsymbol{u}, \forall \boldsymbol{u} \in V$,

(A_5) $-\boldsymbol{u} + \boldsymbol{u} = \boldsymbol{0} = \boldsymbol{u} + (-\boldsymbol{u}), \forall \boldsymbol{u} \in V$,

and the axioms for scalar multiplication

(S_1) *If* $\boldsymbol{u} \in V$, *then* $a\boldsymbol{u} \in V, \forall a \in \mathbb{C}$,

(S_2) $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}, \forall \boldsymbol{u}, \boldsymbol{v} \in V \wedge \forall a \in \mathbb{C}$,

(S_3) $(a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}, \forall \boldsymbol{u} \in V \wedge \forall a,b \in \mathbb{C}$,

(S_4) $a(b\boldsymbol{u}) = (ab\boldsymbol{u}), \forall \boldsymbol{u} \in V \wedge \forall a,b \in \mathbb{C}$,

(S_5) $1\boldsymbol{u} = \boldsymbol{u}, \forall \boldsymbol{u} \in V$.

Remark 5. *The content of axioms* (A_1) *and* (S_1) *is described with thes assertion that V is closed under vector addition and scalar multiplication. The element* **0** *in axiom* A_4 *is called the zero vector.*

Definition 6. *If V is a vector space over the field* $\mathbb{C}$, *a subset U of V is called a subspace of V if U itself is a vector space over* $\mathbb{C}$, *where U uses the vector addition and scalar multiplication of V.*

Definition 7. *Let V be a vector space over the field* $\mathbb{C}$, *and let* $\boldsymbol{v}_i \in V$ ($1 \le i \le n$). *Any vector in V of the form* $\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n$, *where* $a_i \in \mathbb{C}$, ($1 \le i \le n$) *is called a linear combination of* v_i, ($1 \le i \le n$).

Definition 8. *The vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ *are said to span or generate V or are said to form a spanning set of V if* $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. *Alternatively,* $\boldsymbol{v}_i \in V$ ($1 \le i \le n$) *span V, if for every vector* $\boldsymbol{v} \in V$ *there exist scalars* $a_i \in \mathbb{C}$ ($1 \le i \le n$) *such that*

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n,$$

i. e., **v** is a linear combination of

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n.$$

Remark 9. *If* $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, *then each vector* $\boldsymbol{v} \in V$ *can be written as a linear combination of the vectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. *Spanning sets have the property that each vector in V has exactly one representation as a linear combinations of these vectors.*

Definition 10. *Let V be a vector space over a field* $\mathbb{C}$. *The vectors* $\boldsymbol{v}_i \in V$ ($1 \le i \le n$) *are said to be linearly independent over* $\mathbb{C}$, *or simply independent, if it satisfies the following condition: if*

$$s_1\boldsymbol{v}_1 + s_2\boldsymbol{v}_2 + \ldots + s_n\boldsymbol{v}_n = 0,$$

then

$$s_1 = s_2 = \ldots = s_n = 0.$$

Otherwise, the vectors that are not linearly independent, are said to be linearly dependent, or simply dependent.

Remark 11. *The trivial linear combination of the vectors* v_i, ($1 \le i \le n$) *is the one with every coefficient zero*

$$0v_1 + 0v_2 + \ldots + 0v_n.$$

Definition 12. *A set of vectors* $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ *is called a basis of V if it satisfies the following two conditions*

1° $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ *are linearly independent*,

2° $V = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$.

Definition 13. *A vector space V is said to be of finite dimension n or to be n-dimensional, written dim V = n, if V contains a basis with n elements.*

Definition 14. *The vector space* $\{\mathbf{0}\}$ *is defined to have dimension* 0.

3 PARADOX APPEARANCE

Chemistry as other natural sciences is not immune of paradoxes. Unlike other natural sciences, in chemistry paradoxes appeared some time later, and it has only two, while in other sciences many such contradictions are met. These paradoxes are well-known Levinthal's paradox[17]: *The length of time in which a protein chain finds its folded state is many orders of magnitude shorter than it would be if it freely searched all possible configurations*, and Structure-Activity Relationship (SAR) paradox[18]: *Exceptions to the principle that a small change in a molecule causes a small change in its chemical behavior are frequently profound.*

However, these paradoxes are not alone and there are more, but now another will be mentioned, which appears in the balancing of chemical equations.

In the paper[1], the so-called formal balance numbers (FBN) are introduced, like this: *Formal balance numbers are an aid that may grossly facilitate the problem of balancing complex redox equations. They may be chosen as being equal to the traditional values of oxidation numbers, but not necessarily. An inspection of the redox equation may suggest the optimal values that are to be assigned to formal balance numbers. In most cases, these optimal values ensure that only two elements will 'change their state'* (*i. e. the values of the formal balance numbers*), *allowing the use of the oxidation number technique for balancing equations, in its simplest form. Just like for oxidation numbers, the algebraic sum of the formal balance numbers in a molecule/neutral unit is* 0, *while in an ion it is equal to its charge* (*the sum rule*).

Promptly, it was detected that the procedure given in[1] boils down to using of well-known unconventional oxidation numbers, which previously were advocated by Tóth[19] and Ludwig[20].

Consider this sentence from previous definition: *They may be chosen as being equal to the traditional values of oxidation numbers, but not necessarily.* It is a paradox! If the *formal balance numbers* can be the same as oxidation numbers or not, then the whole definition is illogical. This definition represents only a contradictory premise, which does not have any correlation with balancing chemical equations. The above definition does not speak anything about balancing chemical reactions in a chemical sense of the word, or their solution in a mathematical sense. *In order for a chemical equation to be balanced the first necessary and sufficient condition is its solvability,* but the above definition is far from it.

The so-called *formal balance numbers,* which actually are the same as the well-known oxidation numbers, *are not any criterion for balancing chemical equations.*

If the chemical equation is not formalized, then it generates only paradoxes. To support this assertion some ordinary counterexamples will be given.

1° Balancing chemical equations by using of change in oxidation number procedure has a limited usage! It holds only for some simple equations. *Is it possible to determine valence of elements in some complex organic molecules as they are* $C_{2952}H_{4664}N_{812}O_{832}S_8Fe_4$, $C_{738}H_{1166}N_{812}O_{203}S_2Fe$, *and so on*?

Not yet! For instance, one may show the powerlessness of that procedure by the following two counterexamples.

Example 1.

$$2993\ C_{2952}H_{4664}N_{816}O_{832}S_8Fe_4 + 9300568\ CsClO_3 + 408724\ Li_3PO_4 \rightarrow 11972\ Li_4Fe(CN)_6 + 8763504CO_2 + 2370456CsNO_3 + 1178284\ LiHCl_2 + 408724\ H_3PO_4 + 6944000\ Cs_{0.998}Cl + 23944\ H_2S + 5753504\ H_2O.$$

What is the valence of C, N, S *and* Fe?

Example 2.

$$249000\ C_{738}H_{1166}N_{812}O_{203}S_2Fe + 1098133767\ Pb(NO_3)_2 + 46812000\ HSiCl_3 + 7716178\ P_2O_5 + 23148534\ CoCl_2 \rightarrow 46812000\ SiCl_{3.989} + 7716178\ Co_3(PO_4)_2 + 124500\ H_2SO_4 + 549616500\ Pb_{1.998}O_3 + 124500\ Fe_2(SO_4)_3 + 15313500\ C_{12}H_{22}O_{11} + 2398455534\ NO_2.$$

The same previous question holds for this counterexample too. This means, the solution of the above equations is based on very sophisticated matrix methods[21–23], but in no case on the change in oxidation number procedure!

As consequence of that, same holds for so-called *formal balance numbers*. That procedure is useless to balance complex chemical equations as it does not give effects for balancing chemical equations. Still, there are many causes for arguing why that elementary procedure is useless, but the above mentioned two counterexamples are enough to show that it is ineffective.

2° Consider this simple reaction

Example 3.

$$x_1\ NaNO_2 + x_2\ FeSO_4 + x_3\ H_2SO_4 \rightarrow x_4\ NaHSO_4 + x_5\ Fe_2(SO_4)_3 + x_6\ NO.$$

In the above chemical reaction N and Fe change the valence, but chemical equation is impossible! Thus, is the procedure of *formal balance numbers* useful for balancing all chemical equations? Obviously, not! Since it does not have the capability to detect if some chemical equation is possible or not.

3° Now, one more counterexample will be given, where that particular procedure of *formal balance numbers* is impossible.

For instance, consider this chemical reaction

Example 4.

$$128147987406\ FeSO_4 + 27736706439\ PrTlTe_3 + 286573109604\ H_3PO_4 \rightarrow 128404797000\ Fe_{0.998}(H_2PO_4)_2{\cdot}H_2O + 13945051000\ Tl_{1.989}(SO_3)_3 + 13889187000\ Pr_{1.997}(SO_4)_3 + 83210119317\ TeO_2 + 14881757802\ P_2O_3 + 44645273406\ H_2S.$$

Is it possible to balance the above chemical equation by the procedure of *formal balance numbers*?

Not yet! Since it is not possible to determine the valence of Fe, Tl and Pr. Then, on what basis the author of the paper[1] states, that *his* procedure (there called *method*) *is proposed for fast and easy balancing of complex redox equations*? On top of all, he states that *the procedure* (there called *method*) *is probably the fastest of all possible methods*! Really, a *very modest* statement is offered by the author, which is wrong, not only because of today's current balancing methods view point, but also from an earlier view point, when that procedure was published.

Is it a *method* when somebody can find on every step counterexamples? Obviously, the answer is negative! It is merely a picture of the old chemical traditionalism. Or, maybe it is a lonely case lost in the newest progressive and contemporary mathematical generalism!

4 PARADOX RESOLUTION BY A NEW COMPLEX VECTOR METHOD

With the purpose of solution of the paradox, in this section a new complex vector method of balancing chemical equations will be developed. This method is founded on the theory of n-dimensional complex vector spaces.

Theorem 15. *Suppose that chemical equation*

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_n\boldsymbol{v}_n = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{v}_i$ $(1 \le i \le n)$ *are the molecules and* x_i $(1 \le i \le n)$ *are unknown coefficients is a vector space* V *over the field* $\mathbb{C}$ *spanned by the vectors of the molecules* v_i $(1 \le i \le n)$. *If any set of* m *vectors of the molecules in* V *is linearly independent, then* $m \le n$.

Proof. Let be $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. We must show that every set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$ of vectors in V with $m > n$ fails to be linearly independent. This is accomplished by showing that numbers $x_1, x_2, \ldots, x_m$ can be found, not all zero, such that

$$x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + \ldots + x_m\boldsymbol{u}_m = 0.$$

Since V is spanned by the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, each vector $\boldsymbol{u}_j$ can be expressed as a linear combination of $\boldsymbol{v}_i$

$$\boldsymbol{u}_j = a_{1j}\boldsymbol{v}_1 + a_{2j}\boldsymbol{v}_2 + \ldots + a_{nj}\boldsymbol{v}_n.$$

Substituting these expressions into the preceding equation gives

$$\boldsymbol{0} = \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} a_{ij}\boldsymbol{v}_j \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij}x_j \right) \boldsymbol{v}_i$$

This is certainly the case if each coefficient of v_i is zero, *i. e.*, if

$$\sum_{j=1}^{m} a_{ij}x_j = 0, \ (1 \le i \le n).$$

This is a system of n equations with m variables x_1, $x_2, \ldots, x_m$, and because $m > n$, it has a nontrivial solution.

This is what we wanted. Now we shall prove the following results.

Theorem 16. *Let U be a subset of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*. Then U is a subspace of V if and only if it satisfies the following conditions*

$$\mathbf{0} \in U, \text{ where } \mathbf{0} \text{ is the zero vector of } V, \tag{2}$$

$$\text{If } \boldsymbol{u}_1, \boldsymbol{u}_2 \in U, \text{ then } (\boldsymbol{u}_1 + \boldsymbol{u}_2) \in U, \tag{3}$$

$$\text{If } \boldsymbol{u} \in U, \text{ then } a\boldsymbol{u} \in U, \forall a \in \mathbb{C}. \tag{4}$$

Proof. If U is a subspace of V of the chemical equation (1), it is clear by axioms (A_1) and (S_1) that the sum of two vectors in U is again in U and that any scalar multiple of a vector in U is again in U. In other words, U is *closed* under the vector addition and scalar multiplication of V. The converse is also true, *i. e.*, if U is closed under these operations, then all the other axioms are automatically satisfied. For instance, axiom (A_2) asserts that holds $\boldsymbol{u}_1 + \boldsymbol{u}_2 = \boldsymbol{u}_2 + \boldsymbol{u}_1$, $\forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in U$. This is clear because the equation is already true in V, and U uses the same addition as V. Similarly, axioms (A_3), (S_2), (S_3), (S_4) and (S_5) hold automatically in U, because they are true in V. All that remains is to verify axioms (A_4) and (A_5).

If (2), (3) and (4) hold, then axiom (A_4) follows from (2) and axiom (A_5) follows from (4), because $-\boldsymbol{u} = (-1)\boldsymbol{u}$ lies in U, $\forall \boldsymbol{u} \in U$. Hence, U is a subspace by the above discussion. Conversely, if U is a subspace, it is closed under addition and scalar multiplication and this gives (3) and (4). If z denotes the zero vector of U, then $\boldsymbol{z} = 0\boldsymbol{z}$ in U. But, $0\boldsymbol{z} = \mathbf{0}$ in V, so $\mathbf{0} = \boldsymbol{z}$ lies in U. This proves (2).

Remark 17. *If U is a subspace of V of the chemical equation* (1) *over the field, then the above proof shows that U and V share the same zero vector. Also, if* $\boldsymbol{u} \in U$*, then* $-\boldsymbol{u} = (-1)\boldsymbol{u} \in U$*, i. e., the negative of a vector in U is the same as its negative in V.*

Proposition 18. *If V is any vector space of the chemical equation* (1) *over the field* $\mathbb{C}$*, then* $\{\mathbf{0}\}$ *and V are subspaces of V.*

Proof. $U = V$ clearly satisfies the conditions of the Theorem 16. As to $U = \{\mathbf{0}\}$, it satisfies the conditions because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$, $\forall a \in \mathbb{C}$.

Remark 19. *The vector space* $\{\mathbf{0}\}$ *is called the zero subspace of V of the chemical equation* (1) *over the field* $\mathbb{C}$*. Since all zero subspaces look alike, we speak of the zero vector space and denote it by* 0*. It is the unique vector space containing just one vector.*

Proposition 20. *If* $\boldsymbol{v}$ *is a vector of some molecule in a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*, then the set* $\mathbb{C}\boldsymbol{v} = \{a\boldsymbol{v}, \forall a \in \mathbb{C}\}$ *of all scalar multiplies of* $\boldsymbol{v}$ *is a subspace of V.*

Proof. Since $\mathbf{0} = 0\boldsymbol{v}$, it is clear that $\mathbf{0}$ lies in $\mathbb{C}\boldsymbol{v}$. Given two vectors $a\boldsymbol{v}$ and $b\boldsymbol{v}$ in $\mathbb{C}\boldsymbol{v}$, their sum $a\boldsymbol{v} + b\boldsymbol{v} = (a + b)\boldsymbol{v}$ is also a scalar multiple of $\boldsymbol{v}$ and so lies in $\mathbb{C}\boldsymbol{v}$. Therefore $\mathbb{C}\boldsymbol{v}$ is closed under addition. Finally, given $a\boldsymbol{v}$, $r(a\boldsymbol{v}) = (ra)\boldsymbol{v}$ lies in $\mathbb{C}\boldsymbol{v}$, so $\mathbb{C}\boldsymbol{v}$ is closed under scalar multiplication. Now, if we take into account the Theorem 16, immediately follows the statement of the proposition.

Theorem 21. *Let* $U = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *in a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*. Then,*

U is a subspace of V containing each of $\boldsymbol{v}_i$ $(1 \le i \le n)$,(5)

U is the smallest subspace containing these vectors in the sense that any subspace of V that contains each of v_i $(1 \le i \le n)$*, must contain U.*(6)

Proof. First we shall proof (5). Clearly

$$\mathbf{0} = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \ldots + 0\boldsymbol{v}_n$$

belongs to U. If

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_n\boldsymbol{v}_n$$

and

$$\boldsymbol{w} = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + \ldots + b_n\boldsymbol{v}_n$$

are two members of U and $a \in U$, then

$$\begin{aligned} &\boldsymbol{v} + \boldsymbol{w} \\ &= (a_1 + b_1)\boldsymbol{v}_1 + (a_2 + b_2)\boldsymbol{v}_2 + \ldots + (a_n + b_n)\boldsymbol{v}_n, \\ &a\boldsymbol{v} = (aa_1)\boldsymbol{v}_1 + (aa_2)\boldsymbol{v}_2 + \ldots + (aa_n)\boldsymbol{v}_n, \end{aligned}$$

so both $\boldsymbol{v} + \boldsymbol{w}$ and $a\boldsymbol{v}$ lie in U. Hence, U is a subspace of V. It contains each of $\boldsymbol{v}_i$ $(1 \le i \le n)$. For instance,

$$\boldsymbol{v}_2 = 0\boldsymbol{v}_1 + 1\boldsymbol{v}_2 + 0\boldsymbol{v}_3 + \ldots + 0\boldsymbol{v}_n.$$

This proves (5).

Now, we shall prove (6). Let W be subspace of V that contains each of $\boldsymbol{v}_i$, $(1 \le i \le n)$. Since W is closed under scalar multiplication, each of $a_i\boldsymbol{v}_i$ $(1 \le i \le n)$ lies in W for any choice of a_i $(1 \le i \le n)$ in $\mathbb{C}$. But, then $a_i\boldsymbol{v}_i$ $(1 \le i \le n)$ lies in W, because W is closed under addition. This means that W contains every member of U, which proves (6).

Theorem 22. *The intersection of any number of subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$ *is a subspace of V.*

Proof. Let $\{W_i: i \in \boldsymbol{I}\}$ be a collection of subspaces of V and let $W = \cap\, (W_i: i \in \boldsymbol{I})$. Since each W_i is a subspace, then $\mathbf{0} \in W_i$, $\forall i \in \boldsymbol{I}$. Thus $\mathbf{0} \in W$. Assume $u, v \in W$. Then, $\boldsymbol{u}, \boldsymbol{v} \in W_i$, $\forall i \in \boldsymbol{I}$. Since each W_i is a subspace, then $(a\boldsymbol{u} + b\boldsymbol{v}) \in W_i$, $\forall i \in I$. Therefore $(au + bv) \in W$. Thus W is a subspace of V of the chemical equation (1).

Theorem 23. *The union* $W_1 \cup W_2$ *of subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$ *need not be a subspace of V.*

Proof. Let $V = \mathbb{C}^2$ and let $W_1 = \{(a, 0): a \in \mathbb{C}\}$ and $W_2 = \{(0, b): b \in \mathbb{C}\}$. That is, W_1 is the x-axis and W_2 is the y-axis in $\mathbb{C}^2$. Then W_1 and W_2 are subspaces of V of the chemical equation (1). Let $\boldsymbol{u} = (1, 0)$ and $\boldsymbol{v} = (0, 1)$. Then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ both belong to the union $W_1 \cup W_2$, but $\boldsymbol{u} + \boldsymbol{v} = (1, 1)$ does not belong to $W_1 \cup W_2$. Thus $W_1 \cup W_2$ is not a subspace of V.

Theorem 24. *The homogeneous system of linear equations obtained from the chemical equation* (1)*, in n* unknowns $x_1, x_2, \ldots, x_n$ *over the field* $\mathbb{C}$ *has a solution set* *W, which is a subspace of the vector space* $\mathbb{C}^n$.

Proof. The system is equivalent to the matrix equation $\boldsymbol{Ax} = \mathbf{0}$. Since $\boldsymbol{A}\mathbf{0} = \mathbf{0}$, the zero vector $\mathbf{0} \in W$. Assume $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in W, *i. e.*, $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of the matrix equation $\boldsymbol{Ax} = \mathbf{0}$. Then $\boldsymbol{Au} = \mathbf{0}$ and $\boldsymbol{Av} = \mathbf{0}$. Therefore, $\forall a, b \in \mathbb{C}$, we have $\boldsymbol{A}(a\boldsymbol{u} + b\boldsymbol{v}) = a\boldsymbol{Au} + b\boldsymbol{Av} = a\mathbf{0} + b\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Hence $a\boldsymbol{u} + b\boldsymbol{v}$ is a solution of the matrix equation $\boldsymbol{Ax} = \mathbf{0}$, *i. e.*, $a\boldsymbol{u} + b\boldsymbol{v} \in W$. Thus W is a subspace of $\mathbb{C}^n$.

Theorem 25. *If S is a subset of the vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*, then*

1° *the set* span$\{S\}$ *is a subspace of V of the chemical equation* (1) *over the field* $\mathbb{C}$ *which contains S.*

2° span$\{S\} \subseteq W$, *if W is any subspace of V of the chemical equation* (1) *over the field* $\mathbb{C}$ *containing S.*

Proof. 1°. If $S = \varnothing$, then span$\{S\} = \{\mathbf{0}\}$, which is a subspace of V containing the empty set $\varnothing$. Now assume $S \neq \varnothing$. If $\boldsymbol{v} \in S$, then $1v = v \in$ span$\{S\}$, therefore S is a subset of span$\{S\}$. Also, span$\{S\} \neq \varnothing$ because $S \neq \varnothing$. Now assume $\boldsymbol{v}, \boldsymbol{w} \in$ span$\{S\}$; say

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + \ldots + a_m\boldsymbol{v}_m$$

and

$$\boldsymbol{v} = b_1\boldsymbol{w}_1 + \ldots + b_n\boldsymbol{w}_n,$$

where $\boldsymbol{v}_i, \boldsymbol{w}_j \in S$ and a_i, b_j are scalars.

Then

$$\boldsymbol{v} + \boldsymbol{w} = a_1\boldsymbol{v}_1 + \ldots + a_m\boldsymbol{v}_m + b_1\boldsymbol{w}_1 + \ldots + b_n\boldsymbol{w}_n$$

and for any scalar k,
$k\boldsymbol{v} = k(a_1\boldsymbol{v}_1 + \ldots + a_m\boldsymbol{v}_m) = ka_1\boldsymbol{v}_1 + \ldots + ka_m\boldsymbol{v}_m$ belong to span$\{S\}$ because each is a linear combination of vectors in S. Thus span$\{S\}$ is a subspace of V of the chemical equation (1) over the field $\mathbb{C}$ which contains S.

2°. If $S = \varnothing$, then any subspace W contains S, and span$\{S\} = \{\mathbf{0}\}$ is contained in W. Now assume $S \neq \varnothing$ and assume $\boldsymbol{v}_i \in S \subset W$ $(1 \le i \le m)$. Then all multiples $a_i\boldsymbol{v}_i \in W$ $(1 \le i \le m)$ where $a_i \in \mathbb{C}$, and therefore the sum $(a_1\boldsymbol{v}_1 + \ldots + a_m\boldsymbol{v}_m) \in W$. That is, W contains all linear combinations of elements of S. Consequently, span$\{S\} \subseteq W$, as claimed.

Proposition 26. *If W is a subspace of V of the chemical equation* (1) *over the field* $\mathbb{C}$*, then* span$\{W\} = W$.

Proof. Since W is a subspace of V of the chemical equation (1) over the field $\mathbb{C}$, W is closed under linear combinations. Hence, span$\{W\} \subseteq W$. But $W \subseteq$ span$\{W\}$. Both inclusions yield span$\{W\} = W$.

Proposition 27. *If S is a subspace of V of the chemical equation* (1) *over the field* $\mathbb{C}$*, then* span$\{$span$\{S\}\}$ = span$\{S\}$.

Proof. Since span$\{S\}$ is a subspace of V, the above Proposition 26 implies that span$\{$span$\{S\}\}$ = span$\{S\}$.

Proposition 28. *If S and T are subsets of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*, such that* $S \subseteq T$*, then* span$\{S\} \subseteq$ span$\{T\}$.

Proof. Assume $\boldsymbol{v} \in$ span$\{S\}$. Then

$$\boldsymbol{v} = a_1\boldsymbol{u}_1 + \ldots + a_r\boldsymbol{u}_r,$$

where $a_i \in \mathbb{C}$, $(1 \le i \le r)$ and $\boldsymbol{u}_i \in S$ $(1 \le i \le r)$. But $S \subseteq T$, therefore every $\boldsymbol{u}_i \in T$ $(1 \le i \le r)$. Thus $v \in$ span$\{T\}$. Accordingly, span$\{S\} \subseteq$ span$\{T\}$.

Proposition 29. *The* span$\{S\}$ *is the intersection of all the subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$ *which contains S.*

Proof. Let $\{W_i\}$ be the collection of all subspaces of a vector space V of the chemical equation (1) containing S, and let $W = \cap\, W_i$. Since each W_i is a subspace of V, the set W is a subspace of V. Also, since each W_i contains S, the intersection W contains S. Hence span$\{S\} \subseteq W$. On the other hand, span$\{S\}$ is a subspace of V containing S. So span$\{S\} = W_k$ for some k. Then $W \subseteq W_k =$ span$\{S\}$. Both inclusions give span$\{S\} = W$.

Proposition 30. *If* span$\{S\}$ = span$\{S \cup \{\mathbf{0}\}\}$, *then one may delete the zero vector from any spanning set.*

Proof. By Proposition 28, span$\{S\} \subseteq$ span$\{S \cup \{\mathbf{0}\}\}$. Assume $\boldsymbol{v} \in$ span$\{S \cup \{\mathbf{0}\}\}$, say

$$\boldsymbol{v} = a_1\boldsymbol{u}_1 + \ldots + a_n\boldsymbol{u}_n + b\cdot 0$$

where $a_i, b \in \mathbb{C}$ $(1 \le i \le n)$ and $\boldsymbol{u}_i \in S$ $(1 \le i \le n)$.

Then $\boldsymbol{v} = a_1\boldsymbol{u}_1 + \ldots + a_n\boldsymbol{u}_n$, and so $\boldsymbol{v} \in$ span$\{S\}$. Thus span$\{S \cup \{\mathbf{0}\}\} \subseteq$ span$\{S\}$. Both inclusions give span$\{S\}$ = span$\{S \cup \{\mathbf{0}\}\}$.

Proposition 31. *If the vectors* $\boldsymbol{v}_i \in V$ $(1 \le i \le n)$ *span a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*, then for any vector* $\boldsymbol{w} \in V$*, the vectors* $\boldsymbol{w}, \boldsymbol{v}_i$ $(1 \le i \le n)$ *span V.*

Proof. Let $\boldsymbol{v} \in V$. Since the $\boldsymbol{v}_i$ $(1 \le i \le n)$ span V, there exist scalars a_i $(1 \le i \le n)$ such that $\boldsymbol{v} = a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n + 0\boldsymbol{w}$. Thus $\boldsymbol{w}, \boldsymbol{v}_i$ $(1 \le i \le n)$ span V.

Proposition 32. *If* $\boldsymbol{v}_i$ $(1 \le i \le n)$ *span a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*, and for k* > 1*, the vector* $\boldsymbol{v}_k$ *is a linear combination of the preceding vectors* $\boldsymbol{v}_i$ $(1 \le i \le k-1)$*, then* $\boldsymbol{v}_i$ *without* $\boldsymbol{v}_k$ *span V, i. e.,* span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_n\} = V$.

Proof. Let $\boldsymbol{v} \in V$. Since the $\boldsymbol{v}_i$ $(1 \le i \le n)$ span V, there exist scalars a_i $(1 \le i \le n)$ such that

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n.$$

Since $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_i$ $(1 \le i \le k-1)$, there exist scalars b_i $(1 \le i \le k-1)$ such that

$$\boldsymbol{v}_k = b_1\boldsymbol{v}_1 + \ldots + a_{k-1}\boldsymbol{v}_{k-1}.$$

Thus

$$\begin{aligned}\boldsymbol{v} &= a_1\boldsymbol{v}_1 + \ldots + a_k\boldsymbol{v}_k + \ldots + a_n\boldsymbol{v}_n \\ &= a_1\boldsymbol{v}_1 + \ldots + a_k(b_1\boldsymbol{v}_1 + \ldots + b_{k-1}\boldsymbol{v}_{k-1}) + \ldots + a_n\boldsymbol{v}_n \\ &= (a_1 + a_kb_1)\boldsymbol{v}_1 + \ldots + (a_{k-1} + a_kb_{k-1})\boldsymbol{v}_{k-1} \\ &\quad + a_{k+1}\boldsymbol{v}_{k+1} + \ldots + a_n\boldsymbol{v}_n.\end{aligned}$$

Therefore,

$$\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_n\} = V.$$

Proposition 33. *If W_i, $(1 \le i \le k)$ are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *for which* $W_1 \subset W_2 \subset \dots \subset W_k$ *and*

$$W = W_1 \cup W_2 \cup \dots \cup W_k,$$

then W is a subspace of V.

Proof. The zero vector $\mathbf{0} \in W_1$, hence $\mathbf{0} \in W$. Assume $\boldsymbol{u}, \boldsymbol{v} \in W$. Then, there exist j_1 and j_2 such that $\boldsymbol{u} \in Wj_1$ and $\boldsymbol{v} \in Wj_2$. Let $j = \max(j_1, j_2)$. Then $Wj_1 \subseteq W_j$ and $Wj_2 \subseteq W$, and so $\boldsymbol{u}, \boldsymbol{v} \in W_j$. But W_j is a subspace. Therefore $(\boldsymbol{u} + \boldsymbol{v}) \in W_j$ and for any scalar s the multiple $s\boldsymbol{u} \in W_j$. Since $W_j \subseteq W$, we have $(\boldsymbol{u} + \boldsymbol{v})$, $s\boldsymbol{u} \in W$. Thus W is a subspace of V.

Proposition 34. *If W_i $(1 \le i \le k)$ are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$ *and S_i $(1 \le i \le k)$ span W_i $(1 \le i \le k)$, then $S = S_1 \cup S_2 \cup \dots \cup S_k$ spans W.*

Proof. Let $\boldsymbol{v} \in W$. Then there exists j such that $\boldsymbol{v} \in W_j$. Then $\boldsymbol{v} \in \text{span}\{S_j\} \subseteq \text{span}\{S\}$. Therefore $W \subseteq \text{span}\{S\}$. But $S \subseteq W$ and W is a subspace. Hence $\text{span}\{S\} \subseteq W$. Both inclusions give $\text{span}\{S\} = W$, *i. e.*, S spans W.

Theorem 35. *Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be a linearly independent set of vectors in a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *then the following conditions*

1° *$\{\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is a linearly independent set,*

2° *$\boldsymbol{v}$ does not lie in $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$,*

are equivalent for a vector v in V.

Proof. Assume 1° is true and assume, if possible, that $\boldsymbol{v}$ lies in $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$, say,

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + \dots + a_n\boldsymbol{v}_n.$$

Then

$$\boldsymbol{v} - a_1\boldsymbol{v}_1 + \dots + a_n\boldsymbol{v}_n = \mathbf{0}$$

is a nontrivial linear combination, contrary to 1°. So 1° implies 2°. Conversely, assume that 2° holds and assume that

$$a\boldsymbol{v} + a_1\boldsymbol{v}_1 + \dots + a_n\boldsymbol{v}_n = \mathbf{0}.$$

If $a \ne 0$, then

$$\boldsymbol{v} = (-a_1/a)\boldsymbol{v}_1 + \dots + (-a_n/a)\boldsymbol{v}_n,$$

contrary to 2°. So $a = 0$ and

$$a_1\boldsymbol{v}_1 + \dots + a_n\boldsymbol{v}_n = \mathbf{0}.$$

This implies that

$$a_1 = \dots = a_n = 0,$$

because the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is linearly independent. By this is proved that 2° implies 1°.

Proposition 36. *Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be linearly independent in a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *then $\{a_1\boldsymbol{v}_1, a_2\boldsymbol{v}_2, \dots, a_n\boldsymbol{v}_n\}$, such that the numbers a_i $(1 \le i \le n)$ are all nonzero, is also linearly independent.*

Proof. Suppose a linear combination of the new set vanishes

$$s_1(a_1\boldsymbol{v}_1) + s_2(a_2\boldsymbol{v}_2) + \dots + s_n(a_n\boldsymbol{v}_n) = \mathbf{0},$$

where s_i $(1 \le i \le n)$ lie in $\mathbb{C}$.

Then

$$s_1a_1 = s_2a_2 = \dots = s_na_n = 0$$

by the linear independence of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$. The fact that each $a_i \ne 0$ $(1 \le i \le n)$ now implies that $s_1 = s_2 = \dots = s_n = 0$.

Proposition 37. *No linearly independent set of vectors of molecules can contain the zero vector.*

Proof. The set $\{\mathbf{0}, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ cannot be linearly independent, because

$$1\mathbf{0} + 0\boldsymbol{v}_1 + \dots + 0\boldsymbol{v}_n = \mathbf{0},$$

is a nontrivial linear combination that vanishes.

Theorem 38. *A set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ of vectors of molecules in a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$ *is linearly dependent if and only if some $\boldsymbol{v}_i$ is a linear combination of the others.*

Proof. Assume that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is linearly dependent. Then, some nontrivial linear combination vanishes, *i. e.*,

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \dots + a_n\boldsymbol{v}_n = \mathbf{0},$$

where some coefficient is not zero. Suppose $a_1 \ne 0$.

Then

$$\boldsymbol{v}_1 = (-a_2/a_1)\boldsymbol{v}_2 + \dots + (-a_n/a_1)\boldsymbol{v}_n,$$

gives $\boldsymbol{v}_1$ as a linear combination of the others. In general, if $a_i \ne 0$, then a similar argument expresses $\boldsymbol{v}_i$ as linear combination of the others.

Conversely, suppose one of the vectors is a linear combination of the others, *i. e.*,

$$\boldsymbol{v}_1 = a_2\boldsymbol{v}_2 + \dots + a_n\boldsymbol{v}_n.$$

Then, the nontrivial linear combination $1\boldsymbol{v}_1 - a_2\boldsymbol{v}_2 - \dots - a_n\boldsymbol{v}_n$ equals zero, so the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is not linearly independent, *i. e.*, it is linearly dependent. A similar argument works if *any* $\boldsymbol{v}_i$ $(1 \le i \le n)$ is a linear combinations of the others.

Theorem 39. *Let $V \ne 0$ be a vector space of the chemical equation* (1) *over the field* $\mathbb{C}$, *then*

1° *each set of linearly independent vectors is a part of a basis of V,*

2° *each spanning set V contains a basis of V,*

3° *V has a basis and* dim $V \le n$.

Proof. 1° Really, if V is a vector space that is spanned by a finite number of vectors, we claim that *any* linearly independent subset $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ of V is contained in a basis of V. This is certainly true if $V = \text{span}\{S\}$ because then S is *itself* a basis of V. Otherwise, choose $\boldsymbol{v}_{k+1}$ outside $\text{span}\{S\}$. Then $S_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ is linearly independent by Theorem 35. If $V = \text{span}\{S_1\}$, then S_1 is the desired basis containing S. If not, choose $\boldsymbol{v}_{k+2}$ outside $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ so that $S_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2,$

…, $\boldsymbol{v}_k$, $\boldsymbol{v}_{k+1}$, $\boldsymbol{v}_{k+2}\}$ is linearly independent. Continue this process. Either a basis is reached at some stage or, if not, arbitrary large independent sets are found in V. But this later possibility cannot occur by the Theorem 15 because V is spanned by a finite number of vectors.

2° Let V = span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, …, \boldsymbol{v}_m\}$, where (as $V \neq 0$) we may assume that each $\boldsymbol{v}_i \neq \boldsymbol{0}$. If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, …, \boldsymbol{v}_m\}$ is linearly independent, it is itself a basis and we are finished. If not, then according to the Theorem 38, one of these vectors lies in the span of the others. Relabeling if necessary, it is assumed that $\boldsymbol{v}_1$ lies in span$\{\boldsymbol{v}_2, …, \boldsymbol{v}_m\}$ so that V = span$\{\boldsymbol{v}_2, …, \boldsymbol{v}_m\}$. Now repeat the argument. If the set $\{\boldsymbol{v}_2, …, \boldsymbol{v}_m\}$ is linearly independent, we are finished. If not, we have (after possible relabeling) V = span$\{\boldsymbol{v}_3, …, \boldsymbol{v}_m\}$. Continue this process and if a basis is encountered at some stage, we are finished. If not, we ultimately reach V = span$\{\boldsymbol{v}_m\}$. But then $\{\boldsymbol{v}_m\}$ is a basis because $\boldsymbol{v}_m \neq \boldsymbol{0}$ ($V \neq 0$).

3° V has a spanning set of n vectors, one of which is nonzero because $V \neq 0$. Hence 3° follows from 2°.

Corollary 40. *A nonzero vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$ *is finite dimensional only if it can be spanned by finitely many vectors.*

Theorem 41. *Let V be a vector space of the chemical equation* (1) *over the field* $\mathbb{C}$ *and* dim $V = n > 0$, *then*

1° *no set of more than n vectors in V can be linearly independent,*

2° *no set of fewer than n vectors can span V.*

Proof. V can be spanned by n vectors (any basis) so 1° restates the Theorem 15. But the n basis vectors are also linearly independent, so no spanning set can have fewer than n vectors, again by Theorem 15. This gives 2°.

Theorem 42. *Let V be a vector space of the chemical equation* (1) *over the field* $\mathbb{C}$ *and* dim $V = n > 0$, *then*

1° *any set of n linearly independent vectors in V is a basis* (*that is, it necessarily spans V*),

2° *any spanning set of n nonzero vectors in V is a basis* (*that is, they are necessarily linearly independent*).

Proof. 1° If the n independent vectors do not span V, they are a part of a basis of more than n vectors by property 1° of the Theorem 39. This contradicts Theorem 41.

2° If the n vectors in a spanning set are not linearly independent, they contain a basis of fewer than n vectors by property 2° of Theorem 39, contradicting Theorem 41.

Theorem 43. *Let V be a vector space of dimension n of the chemical equation* (1) *over the field* $\mathbb{C}$ *and let U and W denote subspaces of V, then*

1° *U is finite dimensional and* dim $U \leq n$,

2° *any basis of U is a part of a basis for V,*

3° *if* $U \subseteq W$ *and* dim U = dim$W$, *then* $U = W$.

Proof. 1° If $U = 0$, dim$U = 0$ by Definition 14. So assume $U \neq 0$ and choose $\boldsymbol{u}_1 \neq \boldsymbol{0}$ in U. If U = span$\{\boldsymbol{u}_1\}$, then dim$U = 1$. If $U \neq$ span$\{\boldsymbol{u}_1\}$, choose $\boldsymbol{u}_2$ in U outside span$\{\boldsymbol{u}_1\}$. Then $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, is linearly independent by Theorem 35. If U = span$\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, then dim$U = 2$. If not, repeat the process to find $\boldsymbol{u}_3$ in U such that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is linearly independent and continue in this way. The process must terminate because the space V (having dimension n) cannot contain more than n independent vectors. Therefore U has a basis of at most n vectors, proving 1°.

2° This follows from 1° and Theorem 39.

3° Let dim$U$ = dim$W = m$. Then any basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, …, \boldsymbol{u}_m\}$ of U is an independent set of m vectors in W and so is a basis of W by Theorem 42. In particular, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, …, \boldsymbol{u}_m\}$ spans W so, because it also spans U, W = span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, …, \boldsymbol{u}_m\} = U$. By this is proved 3°.

Proposition 44. *If U and W are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *then U + W is a subspace of V.*

Proof. Since U and W are subspaces, $\boldsymbol{0} \in U$ and $\boldsymbol{0} \in W$. Hence $\boldsymbol{0} = \boldsymbol{0} + \boldsymbol{0} \in U + W$. Assume $\boldsymbol{v}, \boldsymbol{v}' \in U + W$. Then there exist $\boldsymbol{u}, \boldsymbol{u}' \in U$ and $\boldsymbol{v}, \boldsymbol{v}' \in W$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ and $\boldsymbol{v}' = \boldsymbol{u}' + \boldsymbol{w}'$. Since U and W are subspaces, $\boldsymbol{u} + \boldsymbol{u}' \in U$ and $\boldsymbol{w} + \boldsymbol{w}' \in W$ and for any scalar k, $k\boldsymbol{u} \in U$ and $k\boldsymbol{w} \in W$. Accordingly, $\boldsymbol{v} + \boldsymbol{v}' = (\boldsymbol{u} + \boldsymbol{w}) + (\boldsymbol{u}' + \boldsymbol{w}') = (\boldsymbol{u} + \boldsymbol{u}') + (\boldsymbol{w} + \boldsymbol{w}') \in U + W$ and for any scalar k, $k\boldsymbol{v} = k(\boldsymbol{u} + \boldsymbol{w}) = k\boldsymbol{u} + k\boldsymbol{w} \in U + W$. Thus $U + W$ is a subspace of V.

Proposition 45. *If U and W are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *then U and W are contained in U + W.*

Proof. Let $\boldsymbol{u} \in U$. By hypothesis W is a subspace of V and so $\boldsymbol{0} \in W$. Hence $\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{0} \in U + W$. Accordingly, U is contained in $U + W$. Similarly, W is contained in $U + W$. By this the proof is finished.

Proposition 46. *If U and W are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *then U + W is the smallest subspace of V containing U and V, i. e.,* $U + W$ = span$\{U, W\}$.

Proof. Since $U + W$ is a subspace of V containing both U and W, it must also contain the linear span of U and W, *i. e.*, span$\{U, W\} \subseteq U + W$.

On the other hand, if $\boldsymbol{v} \in U + W$, then $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} = 1\boldsymbol{u} + 1\boldsymbol{w}$, where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. Hence, $\boldsymbol{v}$ is a linear combination of elements in $U \cup W$ and so belongs to span$\{U, W\}$. Therefore, $U + W \subseteq$ span$\{U, W\}$. Both inclusions give us the required result.

Proposition 47. *If W is a subspace of a vector space V of the chemical equation* (4. 2) *over the field* $\mathbb{C}$, *then W + W = W.*

Proof. Since W is a subspace of V, we have that W is closed under vector addition. Therefore, $W + W \subseteq W$. By Proposition 45, $W \subseteq W + W$. Thus, $W + W = W$. By this the proof is finished.

Proposition 48. *If U and W are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$, *such that* U = span$\{S\}$ *and* W = span$\{T\}$, *then* $U + W$ = *span*$\{S \cup T\}$.

Proof. Since $S \subseteq U \subseteq U + W$ and $T \subseteq W \subseteq U + W$, we have $S \cup T \subseteq U + W$. Hence span$\{S \cup T\} \subseteq U + W$.

Now assume $\boldsymbol{v} \in U + W$. Then $\boldsymbol{v} = \boldsymbol{u} + W$, where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. Since $U = \mathrm{span}\{S\}$ and $W = \mathrm{span}\{T\}$, $\boldsymbol{u} = a_1\boldsymbol{u}_1 + \ldots + a_r\boldsymbol{u}_r$ and $\boldsymbol{w} = b_1\boldsymbol{w}_1 + \ldots + b_s\boldsymbol{w}_s$, where $a_i, b_j \in \mathbb{C}$, $\boldsymbol{u}_j \in S$, and $\boldsymbol{w}_i \in T$. Then $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} = a_1\boldsymbol{u}_1 + \ldots + a_r\boldsymbol{u}_r + b_1\boldsymbol{w}_1 + \ldots + b_s\boldsymbol{w}_s$. Thus, $U + W \subseteq \mathrm{span}\{S \cup T\}$. Both inclusions yield $U + W = \mathrm{span}\{S \cup T\}$.

Proposition 49. *If U and W are subspaces of a vector space V of the chemical equation* (1) *over the field* $\mathbb{C}$*, then* $V = U + W$ *if every* $\boldsymbol{v} \in V$ *can be written in the form* $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$*, where* $\boldsymbol{u} \in U$ *and* $\boldsymbol{w} \in W$.

Proof. Assume, for any $\boldsymbol{v} \in V$, we have $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. Then $\boldsymbol{v} \in U + W$ and so $V \subseteq U + W$. Since U and V are subspaces of V, we have $U + W \subseteq V$. Both inclusions imply $V = U + W$.

By this, in whole is given the skeleton of the complex vector method.

5 AN APPLICATION OF THE MAIN RESULTS

In this section the above complex vector method will be applied on some chemical equations for their balancing. All chemical equations balanced here appear for the first time in professional literature and they are chosen with an intention to avoid all well-known to date chemical equations which were repeated many times in the chemical journals for explanation of certain particular techniques for balancing of some chemical equations using only atoms with integer oxidation numbers.

1° First, we shall consider the case when the chemical reaction is infeasible.

Example 5. Consider this chemical reaction

$$z_1\,{}_{K_4}Fe(CN)_6 + z_2\,{}_{K_2}S_2O_3 \rightarrow z_3\,CO_2 + z_4\,K_2SO_4 + z_5\,NO_2 + z_6\,FeS,$$
$$z_i \in \mathbb{C},\ (1 \leq i \leq 6).$$

From the following scheme

	$v_1 = K_4Fe(CN)_6$	$v_2 = K_2S_2O_3$	$v_3 = CO_2$	$v_4 = K_2SO_4$	$v_5 = NO_2$	$v_6 = FeS$
K	4	2	0	2	0	0
Fe	1	0	0	0	0	1
C	6	0	1	0	0	0
N	6	0	0	0	1	0
S	0	2	0	1	0	1
O	0	3	2	4	2	0

follows this vector equation

$$z_1v_1 + z_2v_2 = z_3v_3 + z_4v_4 + z_5v_5 + z_6v_6,$$

i. e.,

$$z_1\,(4, 1, 6, 6, 0, 0)^T + z_2\,(2, 0, 0, 0, 2, 3)^T = z_3\,(0, 0, 1, 0, 0, 2)^T + z_4\,(2, 0, 0, 0, 1, 4)^T + z_5\,(0, 0, 0, 1, 0, 2)^T + z_6\,(0, 1, 0, 0, 1, 0)^T,$$

or

$$(4z_1 + 2z_2,\ z_1,\ 6z_1,\ 6z_1,\ 2z_2,\ 3z_2)^T = (2z_4,\ z_6,\ z_3,\ z_5,\ z_4 + z_6,\ 2z_3 + 4z_4 + 2z_5)^T.$$

From the system of linear equations

$$4z_1 + 2z_2 = 2z_4,$$
$$z_1 = z_6,$$
$$6z_1 = z_3,$$
$$6z_1 = z_5,$$
$$2z_2 = z_4 + z_6,$$
$$3z_2 = 2z_3 + 4z_4 + 2z_5,$$

one obtains the contradictions $z_2 = 3z_1$ and $z_2 = 44z_1/3$, that means that the system is inconsistent. According to Definition 8, the vectors v_1, v_2, v_3, v_4, v_5 and v_6 of the molecules of the chemical reaction (5. 1) do not generate a vector space V, and according to the Definition 10 they are linearly independent, *i. e.*, we have only a trivial solution $z_i = 0$, $(1 \leq i \leq 6)$, that means that the chemical reaction is infeasible.

2° Next, we shall consider the case when the chemical reaction is feasible and it has a unique solution.

This type of chemical equations really is the most appropriate for study the process of balancing chemical equations, because it gives an excellent opportunity for application of the group theory.

At once, we would like to emphasize here, that by application of groups theory one may determine Sylow subgroups, conjugacy classes of maximal subgroups, proper normal subgroups, and so one. The main reason why we confined ourselves to the next group of calculations is limitation of the size of the work.

Example 6. Consider this chemical reaction

$$z_1\,Fe_2(SO_4)_3 + z_2\,PrTlTe_3 + z_3\,H_3PO_4 \rightarrow z_4\,Fe_{0.996}(H_2PO_4)_2{\cdot}H_2O + z_5\,Tl_{1.987}(SO_3)_3 + z_6\,Pr_{1.998}(SO_4)_3 + z_7\,Te_2O_3 + z_8\,P_2O_5 + z_9\,H_2S,\ z_i \in \mathbb{C},\ (1 \leq i \leq 9).$$

According to the scheme given below

	$v_1 = Fe_2(SO_4)_3$	$v_2 = PrTlTe_3$	$v_3 = H_3PO_4$	$v_4 = Fe_{0.996}(H_2PO_4)_2{\cdot}H_2O$	$v_5 = Tl_{1.987}(SO_3)_3$	$v_6 = Pr_{1.998}(SO_4)_3$	$v_7 = Te_2O_3$	$v_8 = P_2O_5$	$v_9 = H_2S$
Fe	2	0	0	0.996	0	0	0	0	0
S	3	0	0	0	3	3	0	0	1
O	12	0	4	9	9	12	3	5	0
Pr	0	1	0	0	0	1.998	0	0	0
Tl	0	1	0	0	1.987	0	0	0	0
Te	0	3	0	0	0	0	2	0	0
H	0	0	3	6	0	0	0	0	2
P	0	0	1	2	0	0	0	2	0

one obtains the following vector equation

$$z_1v_1 + z_2v_2 + z_3v_3 = z_4v_4 + z_5v_5 + z_6v_6 + z_7v_7 + z_8v_8 + z_9v_9,$$

i. e.,

$$z_1\,(2,3,12,0,0,0,0,0,0)^T + z_2\,(0,0,0,1,1,3,0,0)^T + z_3\,(0,0,4,0,0,0,3,1)^T = z_4\,(0.996,0,9,0,0,0,6,2)^T + z_5\,(0,3,9,0,1.987,0,0,0)^T + z_6\,(0,3,12,1.998,0,0,0,0)^T + z_7\,(0,0,3,0,0,2,0,0)^T + z_8\,(0,0,5,0,0,0,0,2)^T + z_9\,(0,1,0,0,0,0,2,0)^T,$$

from where follows this system of linear equations

$$2z_1 = 0.996z_4,$$
$$3z_1 = 3z_5 + 3z_6 + z_9,$$
$$12z_1 + 4z_3 = 9z_4 + 9z_5 + 12z_6 + 3z_7 + 5z_8,$$
$$z_2 = 1.998z_6,$$
$$z_2 = 1.987z_5,$$
$$3z_2 = 2z_7,$$
$$3z_3 = 6z_4 + 2z_9,$$
$$z_3 = 2z_4 + 2z_8,$$

From the last system one obtains the required solution. This show that the vectors v_i $(1 \le i \le 9)$ generate a vector space V and they are linearly dependent. The balanced reaction has this form

$$17839883133\ Fe_2(SO_4)_3 + 12843034110\ PrTlTe_3 + 81542933266\ H_3PO_4 \rightarrow 35823058500\ Fe_{0.996}(H_2PO_4)_2{\cdot}H_2O + 6463530000\ Tl_{1.987}(SO_3)_3 + 6427945000\ Pr_{1.998}(SO_4)_3 + 19264551165\ Te_2O_3 + 4948408133\ P_2O_5 + 14845224399\ H_2S.$$

3° Next, the case when the chemical reaction is non-unique will be considered, *i. e.*, when it has infinite number of solutions.

Example 7. Consider double reduction of titanium dioxide with carbon and chlorine given by the reaction

$$x_1\ TiO_2 + x_2\ C + x_3\ Cl_2 \rightarrow x_4\ TiCl_4 + x_5\ CO + x_6\ CO_2.$$

This reaction plays an important role in theory of metallurgical processes, especially in processes of direct reduction of metal oxides. Sure, that this reaction is not unique, and there are many other reactions of that kind, which may to be used for analysis of this particular case.

From the following scheme

	$\boldsymbol{v}_1 = TiO_2$	$\boldsymbol{v}_2 = C$	$\boldsymbol{v}_3 = Cl_2$	$\boldsymbol{v}_4 = TiCl_4$	$\boldsymbol{v}_5 = CO$	$\boldsymbol{v}_6 = CO_2$
Ti	1	0	0	1	0	0
O	2	0	0	0	1	2
C	0	1	0	0	1	1
Cl	0	0	2	4	0	0

follows this vector equation

$$z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + z_3\boldsymbol{v}_3 = z_4\boldsymbol{v}_4 + z_5\boldsymbol{v}_5 + z_6\boldsymbol{v}_6,$$

i. e.,

$$(z_1, 2z_1, 0, 0)^T + (0, 0, z_2, 0)^T + (0, 0, 0, 2z_3)^T = (z_4, 0, 0, 4z_4)^T + (0, z_5, z_5, 0)^T + (0, 2z_6, z_6, 0)^T,$$

or

$$(z_1, 2z_1, z_2, 2z_3)^T = (z_4, z_5 + 2z_6, z_5 + z_6, 4z_4)^T,$$

i. e., immediately follows this system of linear equations

$$z_1 = z_4,$$
$$2z_1 = z_5 + 2z_6,$$
$$z_2 = z_5 + z_6,$$
$$2z_3 = 4z_4,$$

which general solution is $z_3 = 2z_1$, $z_4 = z_1$, $z_5 = -2z_1 + 2z_2$, $z_6 = 2z_1 - z_2$, where z_1 and z_2 are arbitrary complex numbers.

Now, balanced general chemical reaction has this form

$$z_1\ TiO_2 + z_2\ C + 2z_1\ Cl_2 \rightarrow z_1\ TiCl_4 + (-\,2z_1 + 2z_2)\ CO + (2z_1 - z_2)\ CO_2, \quad (\forall z_1, z_2 \in \mathbb{C}).$$

According to the Definition 8, the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$, $\boldsymbol{v}_5$ and $\boldsymbol{v}_6$ of the molecules of the chemical reaction generate infinite number of vector spaces V_∞, and according to the Definition 10 they are linearly dependent, *i. e.*, we have an infinite number of solutions $(z_1, z_2, 2z_1, z_1, -\,2z_1 + 2z_2, 2z_1 - z_2)$, $(\forall z_1, z_2 \in \mathbb{C})$, that means that the this chemical reaction is non-unique.

6 DISCUSSION

In his previous work[22], the author announced a decrease of barren intuitionism from of chemistry and its substitution by an elegant formalism from one side, and substitution of the old chemical traditionalism by a new mathematical generalism, from other side. This announcement is realized in this work that gives a new contribution to the theory as well as practice of balancing chemical equations.

The complex vector method of balancing chemical equations augmented the research field in chemistry and made obsolete the old traditional approach, and gave reliable results for paradox resolution.

By this work will begin consideration of paradoxes in chemistry as a serious object, and it will increase researchers' carefulness to avoid appearance of paradoxes.

7 CONCLUSION

The new complex mathematical method of balancing chemical equations, which was used for the solution of a paradox is farewell to the chemical tradition, which still respects the composers of *general chemistry* textbooks, that affirm that chemistry is a science which studies the structure of substances, how they react when combined or in contact, and how they behave under different conditions. These subjects include a great part of the matter to which chemistry was applied.

In this work the *foundation of chemistry* is enriched by one more new topic, and a contribution to a new formalization of chemistry founded by virtue of a new complex vector method of balancing chemical equations is offered. This work opens doors for the next research in chemistry for diagnostic of paradoxes and their resolution. It will accelerate the newest mathematical research in chemistry and it will surmount the barriers hampering the development of chemistry.

This work is a critical survey that requires changes of chemical thinking. Hence, it must be distinguished from the uncritical penetration, in which chemistry itself is developed sometimes.

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8 REFERENCES

[1] Petruševski, V. M.; *Glas. hem. tehnol. Makedonija* 17 (**1998**), 141
[2] Simons, J. H.; *J. Chem. Educ.,* 3 (**1926**), 1305
[3] Ten Hoor, M. J.; *J. Chem. Educ.*, 74 (**1997**), 1367
[4] Herndon, W. C.; *J. Chem. Educ.*, 74 (**1997**), 1359
[5] Bottomley, J.; *Chem. News* 37 (**1878**), 110
[6] Johnson, O. C.; *Chem. News* 42 (**1880**), 51
[7] Waldbauer, L. J.; Thrun, W. E.; *J. Chem. Educ. 3* (**1926**), 1430
[8] Jones, M.; *SIAM Rev.* 13 (**1971**), 571
[9] Alberty, R. A.; *J. Chem. Educ., 68* (**1991**), 984
[10] Zeggeren, V. F.; Storey, S. H.; *The Computation of Chemical Equlibria,* Cambridge Univ. Press: London, 1970
[11] Smith, W. R.; Missen, R. W.; *Chemical Reaction Equilibrium Analysis*: *Theory and Algorithms,* Wiley: New York, 1982
[12] Risteski, I. B.; *SIAM Problems & Solutions*, 2007, 1–10
[13] Risteski, I. B.; *J. Korean Chem. Soc., 52* (**2008**), 223
[14] Crocker, R.; *J. Chem. Educ., 45* (**1968**), 731
[15] Halmos, P.; *Finite Dimensional Vector Spaces,* Springer Vlg: Berlin, 1998
[16] Kurepa, S.; *Konačno dimezionalni vektorski prostori i primjene,* 5-to izd., Sveučilisna naklada Liber: Zagreb, 1990
[17] Levinthal, C.; *J. Chim. Phys., 65* (**1968**), 44
[18] Patani, G. A.; LaVoie, E. J.; Chem. Rev., 96 (**1996**), 3147
[19] Tóth, Z.; *J. Chem. Educ., 74* (**1997**), 1270
[20] Ludwig, O.; *J. Chem. Educ., 74* (**1997**), 1270
[21] Risteski, I. B.; *Internat. J. Math. Manuscripts* 1 (**2007**), 180
[22] Risteski, I. B.; *Bol. Soc. Quím. Méx.* 2 (**2008**), 104
[23] Risteski, I. B.; *J. Chin. Chem. Soc., 56* (**2009**), 65