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A NEW SOLUTION OF THE HARMONIC FUNCTIONS IN THE THEORY OF ELASTICITY

NEKATERE ZNAČILNOSTI HARMONIČNIH FUNKCIJ V TEORIJI O ELASTNIČNOSTI

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A new approach to the solution of a plane problem of the theory of elasticity with the use of two harmonic functions with a Cauchy-Riemann analytical link is developed. The analysis of the harmonic functions shows that some allow a new approach to the solution of problems of the theory of elasticity. For the solution of linear differential equations a fundamental substitution is used, written in the general form $\psi(x,y) = y = C_\sigma \cdot \exp\theta$, with $\theta = \theta(x,y)$ as a function of the strain centre.

The transformations are explained with the properties of harmonic functions, where the Cauchy-Riemann relations can be used. The considered variants extend the possibilities for solutions and, if necessary, to obtain suitable functions for predetermined tasks. The new method is universal and can be effectively used when the fields of stresses and strains are described with trigonometric expressions.

Key words: theory of elasticity, harmonic functions, Cauchy-Riemann expressions.

Razvit je bil nov način reševanja ravninskega problema teorije elastičnosti s Cauchy-Riemanovo analitično zvezo. Analiza harmoničnih funkcij pokaže, da nekatere omogočajo nov način za rešitve problemov iz teorije elastičnosti. Za rešitev linearnih diferencialnih enačb se uporablja temeljna substitucija, zapisana v splošni obliki z $\psi(x,y) = y = C_\sigma \cdot \exp\theta$, z $\theta = \theta(x,y)$ kot funkcijo središča deformacije.

Transformacije smo razložili z lastnostmi harmoničnih funkcij, pri katerih je dovoljena uporaba Cauchy-Riemanovih povezav. Upoštevane variante razširjajo možnost rešitev in, če je potrebno, omogočijo, da dobimo rešitve za vnaprej načrtovano uporabo. Nova metoda je univerzalna in se lahko učinkovito uporabi, če so polja napetosti in deformacij opisana s trigonometričnimi odvisnostmi.

Ključne besede: teorija elestičnosti, harmonične funkcije, Cauchy-Riemanovi izrazi

1 INTRODUCTION AND FORMULATION OF THE TASK

The analysis of the peculiarities of the harmonic functions shows that some of them allow new approaches to the solution of problems of the theory of elasticity. Let us consider a plane problem of this theory. We have a set of equilibrium equations[1].

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0;\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0 \quad (1)$$

The equation of joint strains

$$\nabla^2(\sigma_x+\sigma_y)=0 \quad (2)$$

The stresses' boundary conditions

$$\tau_n=\frac{\sigma_x-\sigma_y}{2}\cdot\sin 2\alpha-\tau_{xy}\cdot\cos 2\alpha \quad (3)$$

Applying these expressions, the harmonic law of the distribution of contact stresses is determined[2], which formally coincides with that in[3]:

$$\tau_n=-\psi(x,y)\cdot\sin(A\Phi-2\alpha)$$

where $\psi(x,y)$ is the coordinate function of the strain centre; A is the constant determining the elastic state of a deformable medium; Φ is the coordinate function characterizing the allocation of contact shearing stresses; α is the slope angle of an element.

In place of equations (1) and (2), the biharmonic equation (4) can be applied:

$$\nabla^4\varphi=\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial^2 x\cdot\partial^2 y}+\frac{\partial^4\varphi}{\partial y^4} \quad (4)$$

with φ as a stress function.

The expression fulfils the boundary conditions (3)

$$\tau_{xy}=\psi(x,y)\cdot\sin(A\Phi) \quad (5)$$

The stress difference in (3) is determined with

$$\sigma_x-\sigma_y=2\cdot\psi(x,y)\cdot\cos(A\Phi) \quad (6)$$

2 SOLUTION OF THE TASK

The fundamental substitution is often used during the solution of linear differential equations[4], which can be written in the following general form

$$\psi(x,y)=\psi=C_\sigma\cdot\exp\theta \quad (7)$$

with $\theta = \theta(x,y)$ being the unknown coordinate function of the strain centre.

Let us examine the harmonic functions $A\Phi$ and θ. The analytical link between them is admitted by the Cauchy-Riemann expressions [4,5]

$$\theta_x = \pm A\Phi_y \quad \theta_y = \pm A\Phi_x \tag{8}$$

After the derivation of equation (5), consideration of equation (7) and substitution in the equilibrium equations we obtain

$$\frac{\partial\sigma_x}{\partial x} + C_\sigma \cdot \theta_y \cdot \exp\theta \cdot \sin(A\Phi) + $$
$$+ C_\sigma \cdot A\Phi_y \cdot \exp\theta \cdot \cos(A\theta) = 0$$
$$\frac{\partial\sigma_y}{\partial y} + C_\sigma \cdot \theta_x \cdot \exp\theta \cdot \sin(A\Phi) + $$
$$+ C_\sigma \cdot A\Phi_x \cdot \exp\theta \cdot \cos(A\Phi) = 0$$

with θ_y, $A\Phi_x$ as the partial derivatives of the appropriate functions of the coordinates y and x. Passing from one variable to the other with the help of (8), we obtain, after integration and simplifications, the normal and shearing stresses

$$\begin{aligned} \sigma_x &= C_\sigma \cdot \exp\theta \cdot \cos(A\Phi) + f(y) + C \\ \sigma_y &= C_\sigma \cdot \exp\theta \cdot \cos(A\Phi) + f(x) + C \\ \tau_{xy} &= C_\sigma \cdot \exp\theta \cdot \sin(A\Phi) \end{aligned} \tag{9}$$

Substituting $f(y) = f(x) = 0$ in (9), we obtain the relation (6) that fulfils the boundary conditions for equation (3).

Considering (9), the sum of the stresses is

$$\sigma_x + \sigma_y = 2C$$

and the equation of joint strains (2) is automatically fulfilled. It is interesting that during the evaluation of the Laplacian for each value $C_\sigma \cdot \exp\theta \cdot \cos(A\Phi)$ and the substitution (8) the identity $0 \equiv 0$ is obtained. Using this peculiarity, the sum of stresses can be expressed as a product of the functions

$$\sigma' = \sigma_x + \sigma_y = n \cdot C \cdot \exp\theta \cdot \cos(A\Phi) \tag{10}$$

with n as the number that defines the influences of hydrostatic pressure on the medium of the stressed state in the strain zone.

By substituting (10) in (2) we obtain

$$\nabla^2(\sigma_x + \sigma_y) = \nabla^2\left[n \cdot C_\sigma \cdot \exp\theta \cdot \cos(A\Phi)\right] =$$
$$= n \cdot C_\sigma \cdot \exp\theta \Big\{\Big[\theta_{xx} + (\theta_x - A\Phi_y)(\theta_x + A\Phi_y) +$$
$$+ \theta_{yy} + (\theta_y - A\Phi_x)(\theta_y + A\Phi_x)\Big] \cdot \cos(A\Phi) -$$
$$- \Big[2\theta A\Phi_x + 2\theta_y A\Phi_y + A\Phi_{xx} + A\Phi_{yy}\Big] \cdot \sin(A\Phi)\Big\} \tag{11}$$

It is clear from the analysis of the differential equation (11), that it turns to identity under the condition of

$$\theta_x = \pm A\Phi_y \quad \theta_y = \pm A\Phi_x$$

This is the relation (8), which was introduced as an assumption during the solution of the equilibrium equations. Differentiating further, we obtain

$$\theta_{xx} = \pm A\Phi_{yx} \quad \theta_{yy} = \pm A\Phi_{xy}$$
$$\theta_{xy} = \pm A\Phi_{yy} \quad \theta_{yx} = \pm A\Phi_{xx}$$

The last relations convert equation (11) into identity.

The last expressions show that the indicated functions are harmonic, i.e.

$$\theta_{xx} + \theta_{yy} = 0 \quad A\Phi_{xx} + A\Phi_{yy} = 0$$

It is remarkable that the operators of the trigonometrical functions are equal to zero. This peculiarity shows that the function (10) also fulfils the biharmonic equation (4). Considering equation (10), the Laplace of the equation has the form

$$\nabla^2\varphi = \nabla^2\left[C_\sigma \cdot \exp\theta \cdot \cos(A\Phi)\right] =$$
$$= C_\sigma \cdot \exp\theta \cdot \left[\begin{pmatrix} \theta_{xx} + (\theta_x^2 - A\Phi_y^2 + \\ + \theta_{yy} + (\theta_y^2 - A\Phi_x^2 \end{pmatrix} \cdot \cos(A\Phi) - \right.$$
$$\left. - \begin{pmatrix} 2 \cdot \theta_x A\Phi_x + 2\theta_y A\Phi_y + \\ + A\Phi_{xx} + A\Phi_{yy} \end{pmatrix} \cdot \sin(A\Phi)\right] = 0$$

Let us introduce the symbolisms

$$L(x,y) = L = \theta_{xx} + \theta_x^2 - A\Phi_y^2 + \theta_{yy} + \theta_y^2 - A\Phi_x^2$$
$$M(x,y) = M = 2 \cdot \theta_x \cdot A\Phi_x + 2\theta_y \cdot A\Phi_y + A\Phi_{xx} + A\Phi_{yy}$$

Then, with consideration of the symbolisms, the accurate form of the Laplace equation is obtained

$$\alpha(x,y) = \alpha = L \cdot \cos(A\Phi) - M \cdot \sin(A\Phi) = 0$$

If the factors in the trigonometrical functions are equal to zero, also the operators $L = M = 0$. For a more integrated analysis let us write a Laplacian for the function $\alpha(x,y)$

$$\nabla^4\varphi = \frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial^2 x \cdot \partial^2 y} + \frac{\partial^4\varphi}{\partial y^4} =$$
$$= \nabla^4\alpha = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial^2 y}\right) \cdot \big(L \cdot \cos(A\Phi) - M \cdot \sin(A\Phi)\big) =$$
$$= (L_{xx} - L \cdot A\Phi_x^2 - 2M_x \cdot A\Phi_x - M \cdot A\Phi_{xx} + L_{yy} - L \cdot A\Phi_y^2 -$$
$$- 2M_y \cdot A\Phi_y - M \cdot A\Phi_{yy}) \cdot \cos(A\Phi) - (2L_x \cdot A\Phi_x +$$
$$+ L \cdot A\Phi_{xx} + M_{xx} - M \cdot A\Phi_x^2 + 2L_y \cdot A\Phi_y + L \cdot A\Phi_{yy} +$$
$$+ M_{yy} - M_y \cdot A\Phi_y^2) \cdot \sin(A\Phi)$$

It is expected that the partial derivatives from zero functions are equal to zero, thus, $\nabla^4\alpha \equiv 0$. Let us write the partial derivatives of separate operators and track the mechanism of the turning into identity of the harmonic functions

$$L_x = (\theta_{xxx} + \theta_{yyx}) + (2\theta_x\theta_{xx} - 2A\Phi_y A\Phi_{yx} +$$
$$+ (2\theta_y\theta_{yx} - 2A\Phi_x A\Phi_{xx})$$

Following (8), we have

$$\theta_x = -A\Phi_y \quad \theta_y = A\Phi_x$$
$$\theta_{xx} = -A\Phi_{yx} \quad \theta_{yy} = A\Phi_{xy}$$
$$\theta_{xy} = -A\Phi_{yy} \quad \theta_{yx} = A\Phi_{xx}$$
$$\theta_{xxx} = -A\Phi_{yxx} \quad \theta_{yyy} = A\Phi_{xyy}$$

$$\theta_{xyy} = -A\Phi_{yyy} \quad \theta_{yxx} = A\Phi_{xxx} \tag{12}$$

Let us substitute (12) in the operator L_x.

$$L_x = \frac{\partial}{\partial x}(\theta_{xx} + \theta_{yy}) + \left[2\theta_x\theta_{xx} - 2(\theta_x)(\theta_{xx})\right] + $$
$$+\left[2\theta_y\theta_{yx} - 2(\theta_y)(\theta_{yx})\right] \equiv 0$$

We have obtained the identity, "quod erat demonstrandum".

The same approaches take place during the evaluation of the operator L_{xx}. Let us write it as

$$L_{xx} = (\theta_{xxxx} - \theta_{yyxx}) + (2\theta_{xx}\theta_{xx} - 2A\Phi_{yx}A\Phi_{yx}) +$$
$$+(2\theta_x\theta_{xxx} - 2A\Phi_y A\Phi_{yxx}) - (2A\Phi_{xx}A\Phi_{xx} - 2\theta_{yx}\theta_{yx}) -$$
$$-(2A\Phi_x A\Phi_{xxx} - 2\theta_y\theta_{yxx})$$

Substituting (12) into the expression for L_{xx} and factoring out a flexion on x from the first brackets, after conversion the identity $0 \equiv 0$ is obtained.

Thus, the operators

$$L_{xx} = L_{yy} = M_{xx} = M_{yy} = L_x = L_y = M_x = M_y = L = M = 0$$

demonstrate that the function (10) fulfils the biharmonic equation (4) and it can be used for the evaluation of the components of the stress tensor. It is necessary to ensure that the field of stresses and the stress function are described, as a matter of fact, with identical expressions (9) and (10) linked analytically[6] with

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2\varphi}{\partial x^2} \qquad \tau_{xy} = \frac{\partial^2\varphi}{\partial y\partial x}$$

Such schemes of transformations are explained with the properties of harmonic functions where the Cauchy-Riemann relations can be applied. The considered variants allow us to extend the possibility of solutions and, if necessary, to obtain suitable functions for the development of a predetermined result.

Let us return to the expressions for the stress tensor components and consider the equilibrium equations in the components of the stress deviator. Let us introduce the symbols

$$\sigma_x'' = \sigma_x - \sigma - f(y) - C$$
$$\sigma_y'' = \sigma_y - \sigma - f(x) - C \tag{13}$$

with σ being the mean stress.

Considering (13), the equilibrium equation (1), can be rewritten in the form[6]

$$\frac{\partial^2\sigma_x''}{\partial x} + \frac{\partial^2\tau_{xy}}{\partial y} = 0 \qquad \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y''}{\partial y} = 0$$

By analogy with (9) and with integration and simplifications we obtain the stress tensor components

$$\sigma_x = C_\sigma \cdot \exp\theta \cdot \cos(A\theta) + \sigma + f(y) + C$$
$$\sigma_y = -C_\sigma \cdot \exp\theta \cdot \cos(A\theta) + \sigma + f(x) + C$$
$$\tau_{xy} = C_\sigma \cdot \exp\theta \cdot \sin(A\theta) \tag{14}$$

With: $\theta_x = \pm A\Phi_y$, $\theta_y = \pm A\Phi_x$

It follows from expressions (14) that their deviator part for the normal stresses $C_\sigma \cdot \exp\theta \cdot \cos(A\Phi)$ coincides with the shifting part σ in (10). Considering (13), (6) is fulfilled and the boundary conditions (3) are satisfied.

The outcome (14) can be generalized. The analytical link of the functions with the opposite signs is obtained in relations (8) and different signs of an index in an exponential curve result can be obtained. Therefore, the index of an exponential curve in a solution will be not unique. The exponential function can be written in the form of a sum with the use of the hyperbolic cosine or sine in the general form

$$\sigma_x = \left[C_1 \cdot ch(\theta) \pm C_2 \cdot sh(\theta)\right]\cos(A\Phi) + \sigma + f(y) + C$$
$$\sigma_y = -\left[C_1 \cdot ch(\theta) \pm C_2 \cdot sh(\theta)\right]\cos(A\Phi) + \sigma + f(x) + C$$
$$\tau_{xy} = +\left[C_1 \cdot ch(\theta) \pm C_2 \cdot sh(\theta)\right]\sin(A\Phi) \tag{15}$$

In these expressions it is assumed that the arguments of the trigonometric and exponential functions can be represented in the form of a series of harmonic functions interlinked with the Cauchy-Riemann relations.

3 COMPARISON TO OTHER SOLUTIONS

The solutions of a plane problem with the help of a trigonometric series are often used. For example, the following combination of functions is often met[3]:

$$\varphi = \sin(\alpha \cdot x) \cdot \left[C_1 \cdot \exp(\alpha \cdot y) + C_2 \cdot \exp(-\alpha \cdot y)\right] \tag{16}$$

Let us ascertain whether the Cauchy-Riemann relation exists between the arguments of trigonometric and exponential functions

$$A\Phi = \alpha \cdot x \qquad \theta = \pm\alpha \cdot y \qquad A\Phi_x = \alpha$$
$$A\Phi_y = 0 \qquad \theta_y = \pm\alpha \qquad \theta_x = 0$$

The obtained relations take place for the functions

$$\theta_x = \mp A\Phi_y = 0 = 0 \qquad \theta_y = \pm A\Phi_x = \pm\alpha = \alpha$$

The peculiarity of these solutions is that they are common and do not contradict known partial solutions. The arguments $A\Phi$ and θ are harmonic functions of the coordinates x and y. They can be rather complicated and cannot be determined from the linear dependences for one coordinate.

Let us analyze the possibilities of the solution (14). The elementary variant of a harmonic function of two variables is $A\Phi = A \cdot x \cdot y$. Applying the relations (8), it is written as

$$\theta = \mp\frac{1}{2} \cdot A(x^2 - y^2)$$

Thus,

$$\theta_x = \mp A\Phi_y = \mp A \cdot x = A \cdot x \qquad \theta_y = \pm A\Phi_x = \pm A \cdot y = A \cdot y$$

Each function fulfils the Laplace equations.

4 CONCLUSION

A new approach to the solution of a plane problem of the theory of elasticity based on the use of two harmonic functions with the Cauchy-Riemann analytical link is developed. The new method is universal and can be effectively used when the fields of stresses and strains are described with trigonometric expressions.

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